Statistical Models of Turbulent Flow

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(from the SBU Ph. D. thesis of V. Mahadeo)

Topics not included

(recent papers/theses, open for discussion during this visit)

1. Turbulent combustion
2. Turbulent mixing
3. Inertial Confinement Fusion: UQ
4. Inertial Confinement Fusion: fluid transport
5. Short term weather forecasts of cloud cover
6. Cardiac electrophysiology and fibrillation
7. An API for Front Tracking
8. Financial modeling

K41 and K62

Kolmogorov's 1941 scaling law for turbulent kinetic energy is one of the deepest contributions to our (imperfect) understanding of turbulence.

$$< v'v' > \sim \varepsilon^{2/3} k^{-5/3}$$

$$\varepsilon = \text{energy dissipation rate} =$$

$$\frac{1}{2}\nu\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2$$

Mathematically, K41 can be seen as s Sobolev bound, and on the basis of this as a postulate, L_p solutions for the Euler equation can be constructed [CG12]

Kolmogorov 1962 and independently Obkuhov postulated that epsilon is log normal. Pope and Chen assumed a (temporal) log normal stochastic process for epsilon.

$$\chi = \ln \varepsilon$$

A new scaling law
A new universality principal

After adjustment for a (local) mean and time scale, every inertial range velocity gradient degree of freedom is multiplicatively equipartitioned, i.e., contributes equally to the turbulent intensity.

The result is a new scaling law for turbulent intensity:

$$\Sigma = \text{covariance} \ln \varepsilon \sim k^{-2}$$

Epsilon

Epsilon features in many deeper analyses of turbulence

1. Scaling laws for the higher velocity moments
2. Turbulent diffusion
3. Corrections to the Kolmogorov exponent -5/3
4. Clustering of particles in particle laden turbulent flow
5. Short distance asymptotics of the two point correlation function
6. Fractal like intermittency for turbulence: turbulent regions occupy a fractally smaller fraction of space at each smaller length scale.

A Random Field Model for Epsilon

1. Epsilon is log normal-mixture as a random field (dependence on space and time)
2. We do not assume homogeneous isotropic turbulence
3. The theory is thus applicable to Large Eddy Simulations (LES), with resolved deterministic scales and unresolved, stochastic scales.

A Random Field Model for Epsilon, continued

4. The model has been tested (verified) through comparison to Direct Numerical Simulation (DNS) for about a decade of inertial range turbulent flow.
5. Resolved scale flow properties set the parameters of the model.
6. An equipartition hypothesis allows universal modeling with a simple and intuitive parameterization and a new power scaling law.

A Random Field Model for Epsilon, continued

7. Verification through prediction of particle clustering in particle laden flow will be shown.
8. The fractal nature of intermittency does not result from the model.
9. To obtain a fractal solution, the model is revised within a Renormalization Group framework, to decrease the volume of active turbulence on each smaller length scale.

A Random Field Model continued

10. When so revised, the original model serves as a single RNG iteration or integration step.
11. The model universality results from the limited range of turbulent scales modeled, which is sufficient for a single RNG step.

Outline of Presentation

1. A DNS/LES study of turbulence
2. The log normal property
3. Equipartition hypothesis and a scaling law in Fourier space; DNS verification
4. Verification for particle clustering
5. RNG and extensions to multiple unresolved length scales

1. DNS/LES Turbulence

2^3 coarse grid LES cells define a resolved cell, coarse grid LES velocity gradients.
Resolved grid = 2 X coarse grid = 8 X fine grid

Fine grid simulationn	Fine grid	Resolved coarse grid	Re	Taylor Re	Kolmogorov scale
DNS	256^3	16^3	561	40	1.4 Delta
LES	256^3	16^3	1577	67	0.67 Delta
LES	256^3	16^3	2539	85	0.04 Delta

H. Pouransari, H. Kolla, J. H. Chen, A. Mani
Proceedings of Summer program, Center for Turbulence Research, Stanford University, 27-36. 2014.

Random process for χ

Randomness: choice of resolved space time cell
Expectation: sum over resolved cells
Random functions: any subgrid fine grid solution
Apply to ε and χ : functions of fine grid
depending randomly on resolved cell

Remove effects of resolved grid time scale and mean; reduced χ is modeled as universal, with lognormal statistics

2. The log normal property

We define $\chi = \ln \varepsilon$ as turbulent intensity. With ε lognormal, χ is normal. Both considered as random functions of the unresolved scales, depending parametrically on the resolved scales. We propose the stochastic equation

$$\chi(\hat{t}) = \chi(t) - \mu; \quad \hat{t} = t / T$$

$$d\chi(t) = \frac{\chi(t) - \mu}{T} dt + \left(\Sigma^2 / T\right)^{1/2} dW; \quad d\chi = \chi + \Sigma dW$$

dW has independent (white noise) space time increments

Equation Parameters	
χ	$\ln \varepsilon$
μ	Resolved scale mean for χ
Σ^2 / T	Covariance for χ
$T = \Delta_r^2 / \nu_t$	Resolved time scale for χ
η	Resolved scale DNS cutoff for χ
ν_t	Turbulent eddy viscosity at resolved scale

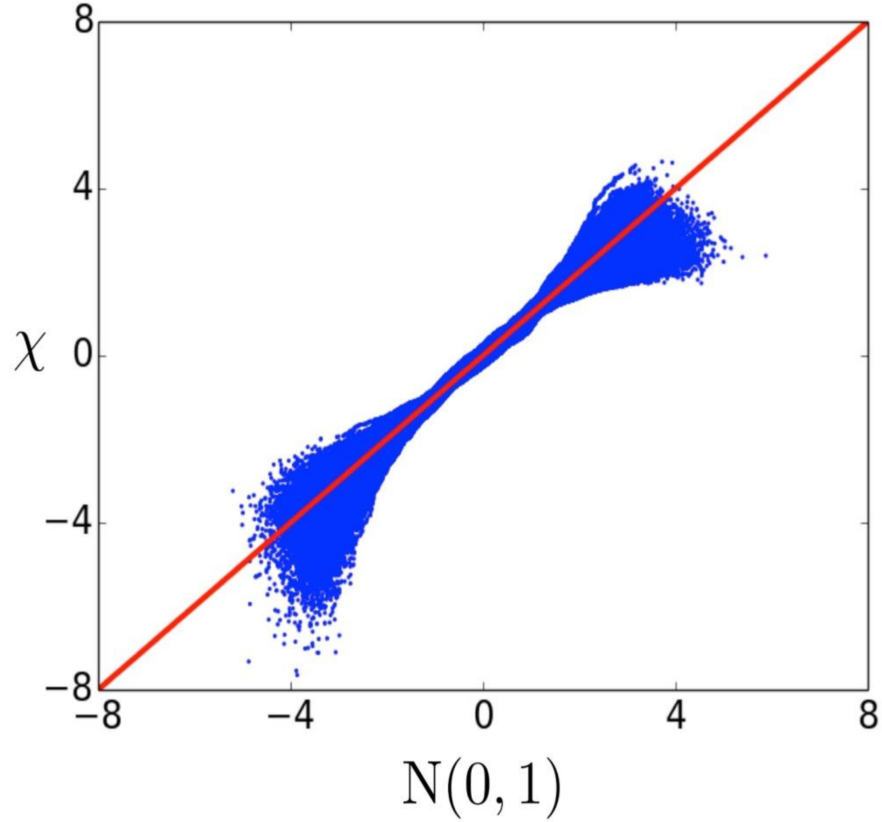
Tests for normality of χ

A multivariate random variable is Gaussian if its inner product with any vector is Gaussian

We choose fine grid mesh values as the test vector.

We use QQ plots to assess the univariate Gaussian property: Transform the test statistic to have unit variance and mean zero, apply an inverse Gaussian change of coordinates and compare to a straight line.

QQ plots for χ
The log normal property



Each resolved cell is plotted separately with all fine grid data points within it defining the PDF of χ All such plots are superimposed here, to show good agreement with the Gaussian property up to +- 2 standard deviations. Integral scale Re = 1577.

3. Universality of subgrid statistics

The purpose of writing the covariance as Σ^2 / T is allow a universal description of Σ

Apply equipartition hypothesis to Σ

4. Equipartition hypothesis and a scaling law in Fourier space

Expand Σ^2 in Fourier space, within a single resolved cell. Assume Σ^2 is diagonal and a constant multiple of the identity for each scalar $k = |\vec{k}|$. Assume each $\vec{k}$ contributes equally to the variance. Then

$$\Sigma^2(k) = \text{const.} \quad k^{-2}$$

$$\text{const. } k^2 = \text{ area of sphere radius } k$$

Corrections to scaling law

1. Corrections for viscous cutoff (eta = Kolmogorov scale)

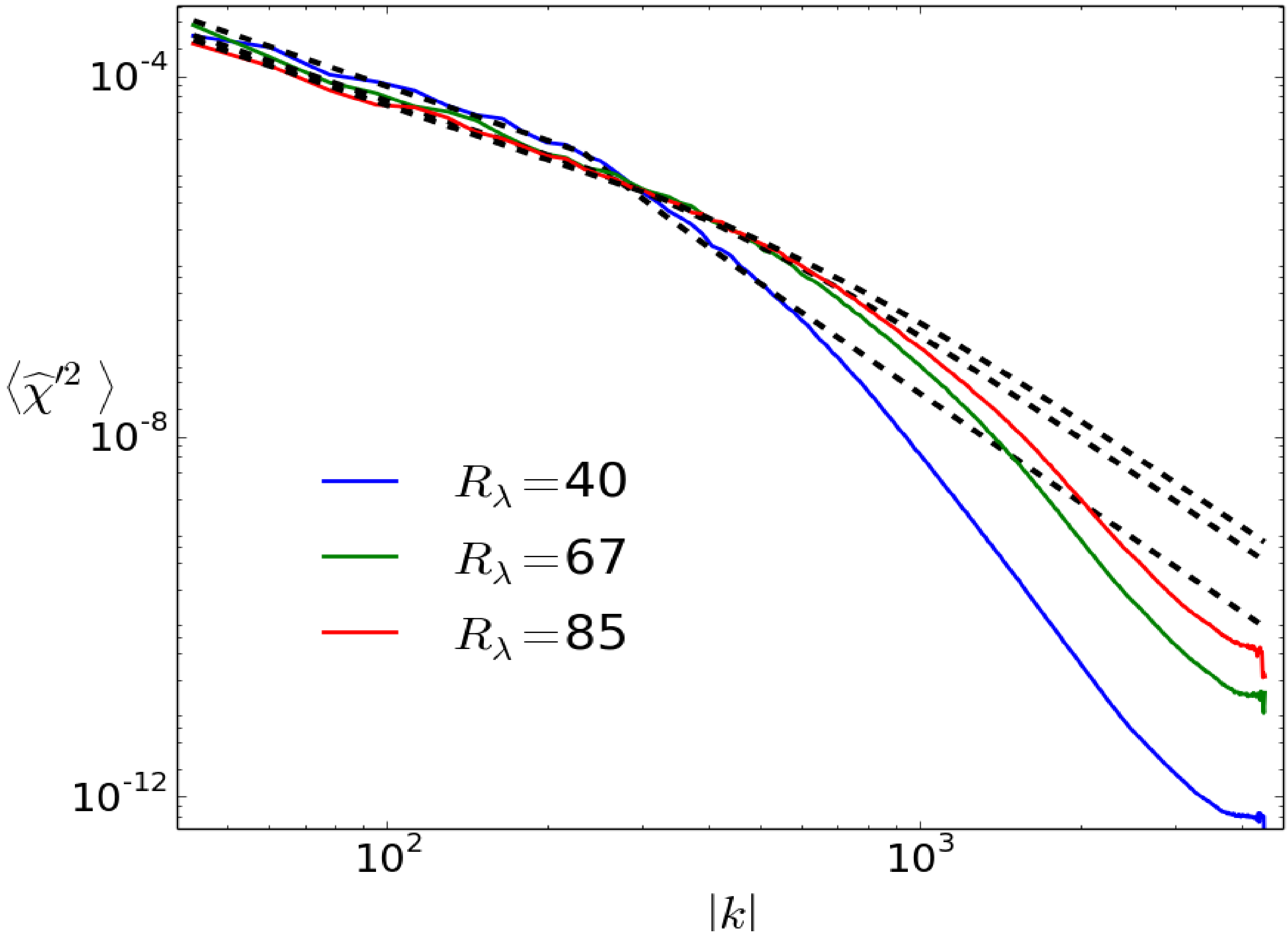
$$\Sigma^2(k) = \left(4\pi n^3 k^2 \left(C_2^2 + C_4^4 \left(\eta k / 2\pi\right)^2\right)\right)^{-1}$$

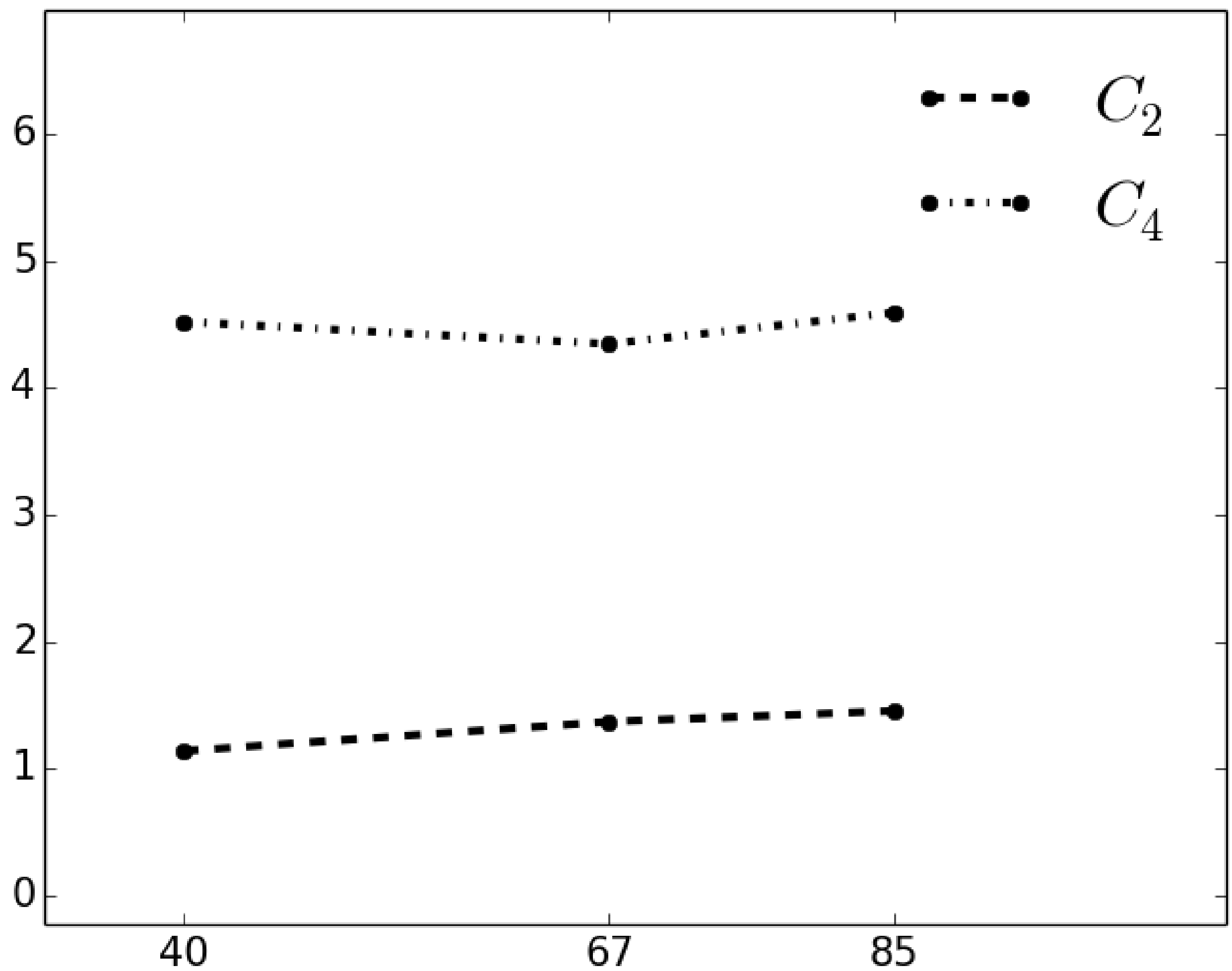
2. Corrections for interaction with finite sized particles.

$$\eta \to \eta_p = \quad \max \quad \left\{1, St\right\}\eta$$

St = ratio of particle to fluid time scale

Model and DNS data compared.
Error for $\int \Sigma(k)dk$: a few percent





C_2
C_4
0
1
2
3
4
5
6
40
67
85

5. Verification for particle clustering

Particle laden flow. Assume small (“point”) particles, low density, nonineracting upon the fluid or with each other. Flow characterized by Stokes number = St

= (fluid-particle equilibration time)/fluid equilibration time

Particle motion from Stokes drag law. But subgrid fluctuations contribute to drag, 1st order in dt by Ito theory.

Ito theory and K62

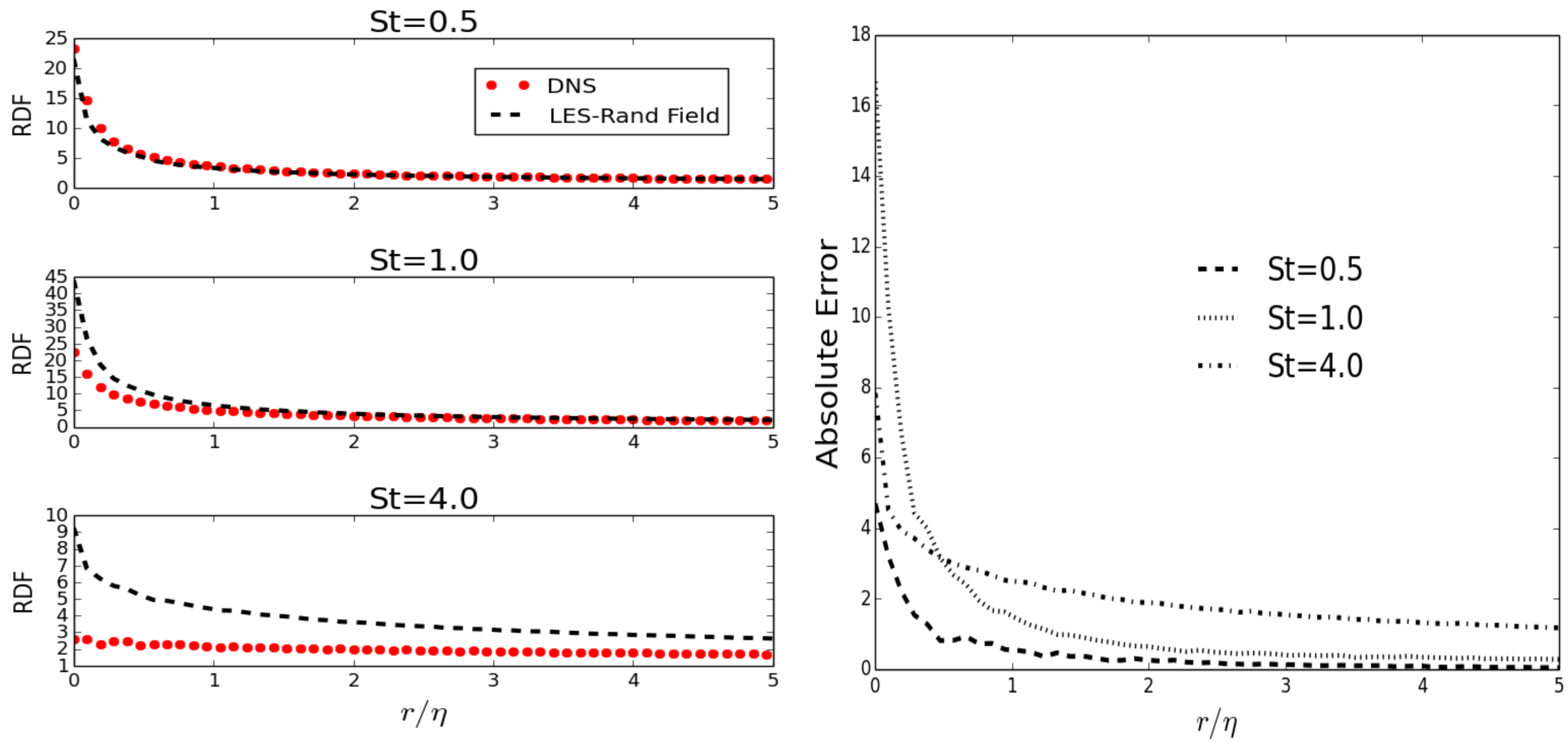
Model longitudinal fluctuations as proportional to ε
Thus log normal (after rescaling). Stochastic SGS model improves clustering property of particles. They cluster in regions of low turbulent intensity.

Directional fluctuations: ongoing work

Measure particle clustering by radial distribution function

$$g(r) = \frac{1}{4\pi r^2} \frac{1}{N_p \rho} \sum_{i=0}^{N_p} \sum_{j=0, j \neq i}^{N_p} \delta(r - r_{i,j})$$

Compare coarse grid + stochastic model to DNS
Left: 3 St numbers; Right: error plot



Discussion

St < = 1 looks good.
St = 4: small inertial range above η_p

For high Re flows, method should be satisfactory (to be confirmed)

6. RNG and many unresolved length scales

The theory presented for turbulence intensity has been tested over about one decade of unresolved scales.

Universality is based on the idea that the resolved scales set length and time dependent parameters for the unresolved scales.

With multiple decades of unresolved scales we have a problem.

The failure of universality

If universality applies across multiple scales, then the information from the resolved scales would not be needed, and the theory would be globally valid for all turbulent flows.

But we see a clear resolved scale dependence in T: A strongly vs. weakly turbulent resolved region influences all its subregions.

Reformulation of Universality

The universal theory for statistics of turbulent intensity is approximately valid over one change of length scales only.

To iterate, and apply to multiple scales, we need to reset model parameters, depending on the larger scales, after which, the theory of the next smaller set of length scales is universal and parameterized.

The resetting of parameters is organized as a group operation (RNG).

Fractal properties of solution

Extreme values of χ (small) will necessarily occur with finite probability. In these regions, the flow will be laminar. Passing to a smaller length scale, this region will have laminar parameters, and should not be log normal.

Thus a finite fraction of the flow region is removed from the turbulent state at each new length scale.

Related to fractal models of turbulence.

Summary Conclusions

1. The single length scale theory is tested and complete.
2. For multiple length scales, the theory can only to applied to the flow region of space that is turbulent for the currently resolved scales.
3. The equations close in the sense that all parameters are functions of epsilon for currently resolved scales. (Use Smagorinsky, not dynamic eddy viscosity.)

Summary, Continued

4. The universal theory gives parameters for the stochastic integration of epsilon for the next smaller set of scales.
5. Monte Carlo (MC) simulation allows solution of the multiple scaled theory.
6. The number of samples does not increase with the number of levels. Rather, a fixed number of MC realizations will suffice for all levels.
7. Mathematical/numerical properties of the multiscale log normal model remain to be explored.